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# Asymmetry of Ext-groups

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we will use techniques of noncommutative projective geometry to construct examples of algebras $R$ over a field $k$ not satisfying the following two types of symmetric behaviors of Ext-groups: (**EE**) For any pair of finitely generated $R$-modules $(M, N)$, $\dim_k \operatorname{Ext}^i_R(M, N) < \infty$ for all $i \in \mathbb{N}$ if and only if $\dim_k \operatorname{Ext}^i_R(N, M) < \infty$ for all $i \in \mathbb{N}$. (**ee**) For any pair of finitely generated $R$-modules $(M, N)$, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(N, M) = 0$ for all $i \gg 0$. In particular, and contrary to the commutative case, we give a simple example of a noncommutative noetherian Gorenstein (Frobenius) local algebra satisfying (**uac**) (uniform Auslander condition) but not satisfying (**ee**).



## 1. Introduction

### 1.1. Motivation

Throughout, we fix a base field $k$. Let $R$ be an algebra over $k$. We denote by $\operatorname{mod} R$ the category of finitely generated right $R$-modules. For $M, N \in \operatorname{mod} R$, we defined in [16] the intersection multiplicity of $M$ and $N$ by $M \cdot N := (-1)^{\operatorname{GKdim} R - \operatorname{GKdim} M}\xi(M, N)$ where $\operatorname{GKdim} M$ is the Gelfand–Kirillov dimension of $M$ and

$$\xi(M, N) := \sum_{i \in \mathbb{N}} (-1)^i \dim_k \operatorname{Ext}^i_R(M, N)$$

is the Euler form of $M$ and $N$. In order for this definition to yield a good intersection theory, we must have the property $M \cdot N = N \cdot M$ for reasonably nice $R, M, N$. Although, for each $i$, $\operatorname{Ext}^i_R(M, N)$ and $\operatorname{Ext}^i_R(N, M)$ are very different in general (even if $R$ is commutative), the property $M \cdot N = N \cdot M$

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holds in many situations. For example, over a commutative noetherian Gorenstein local algebra, this property is equivalent to the Serre's vanishing conjecture [11]. Moreover, for a noetherian connected graded algebra $R$, and finitely generated graded right $R$-modules $M$ and $N$, it is often the case that $M \cdot N = N \cdot M$ as long as both $M \cdot N$ and $N \cdot M$ are well defined [9]. So the natural question is to ask whether $M \cdot N$ is well defined if and only if $N \cdot M$ is well defined (in the graded case). Since $M \cdot N$ is well defined if and only if (i) $\dim_k \operatorname{Ext}^i_R(M,N) < \infty$ for all $i \in \mathbb{N}$, and (ii) $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, we ask whether $R$ satisfies the following two types of symmetric behaviors of Ext-groups:

- (**EE**): for all $M, N \in \operatorname{mod} R$, $\dim_k \operatorname{Ext}^i_R(M,N) < \infty$ for all $i \in \mathbb{N}$ if and only if $\dim_k \operatorname{Ext}^i_R(N,M) < \infty$ for all $i \in \mathbb{N}$.
- (**ee**): for all $M, N \in \operatorname{mod} R$, $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(N,M) = 0$ for all $i \gg 0$.

It is known that every commutative noetherian local algebra satisfies (**EE**) [11, Corollary 3.2]. On the other hand, it is easy to see that if $R$ is a local algebra satisfying (**ee**), then $R$ must be Gorenstein, so we focus on studying Gorenstein algebras in this paper. Presumably, Avramov and Buchweitz were the first people who studied the condition (**ee**), and they proved that every commutative complete intersection ring satisfies (**ee**) in [2]. The first example of a noetherian Gorenstein algebra not satisfying (**ee**) was given by Jorgensen and Sega in [6], which is even a commutative Frobenius local algebra. Later, a simpler but noncommutative (Frobenius) algebra not satisfying (**ee**) was given in [4]. (We thank Petter Andreas Bergh for pointing this out.) In this paper, we will use techniques of noncommutative projective geometry to construct noncommutative algebras not satisfying (**EE**) and those not satisfying (**ee**).

Related to the condition (**ee**), there is another condition (**uac**) (uniform Auslander condition) on a ring $R$:

- (**uac**): there is an integer $d_R \in \mathbb{N}$ such that, for all $M, N \in \operatorname{mod} R$, if $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, then $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > d_R$.

It was shown [5, Theorem 4.1] that every commutative noetherian Gorenstein local ring satisfying (**uac**) satisfies (**ee**). Contrary to the commutative case, we give a simple example of a noncommutative noetherian Gorenstein (Frobenius) local algebra satisfying (**uac**) but not satisfying (**ee**). (In fact, we will see that this happens quite frequently.)

Along the way toward the main results, we also prove that, for every FBN (fully bounded noetherian) AS-Gorenstein Koszul algebra, there is a bijection between isomorphism classes of point modules over $A$ and those over $A^o$, extending [15, Theorem 6.3].

### *1.2. Noncommutative projective geometry*

In this subsection, we review some of the language of noncommutative projective geometry which will be needed in this paper. We refer to [1] for details. Let $A$ be a graded algebra. We denote by $\operatorname{GrMod} A$ the category of graded right $A$-modules, and by $\operatorname{grmod} A$ the full subcategory of finitely generated graded right $A$-modules. Morphisms in $\operatorname{GrMod} A$ are $A$-module homomorphisms preserving degrees. The category of (finitely generated) graded left $A$-modules can be identified with $\operatorname{GrMod} A^o$ ($\operatorname{grmod} A^o$) where $A^o$ is the opposite graded algebra of $A$.

For a vector space $V$ over $k$, we denote by $V^*$ the dual vector space of $V$. For a graded vector space $V \in \operatorname{GrMod} k$, we define the graded vector space dual $V^* \in \operatorname{GrMod} k$ by $(V^*)_i := (V_{-i})^*$ for $i \in \mathbb{Z}$. Moreover, for an integer $n \in \mathbb{Z}$, we define the truncation $V_{\geqslant n} := \bigoplus_{i \geqslant n} V_i \in \operatorname{GrMod} k$ and the shift $V(n) \in \operatorname{GrMod} k$ by $V(n)_i := V_{n+i}$ for $i \in \mathbb{Z}$. We say that $V$ is right bounded if $V_{\geqslant n} = 0$ for some $n \in \mathbb{Z}$. We say that $V$ is locally finite if $\dim_k V_i < \infty$ for all $i \in \mathbb{Z}$. In this case, we define the Hilbert series of $V$ by

$$H_V(t) := \sum_{i \in \mathbb{Z}} (\dim_k V_i) t^i \in \mathbb{Z}[[t, t^{-1}]].$$

For $M, N \in \operatorname{GrMod} A$, we denote by $\operatorname{Hom}_A(M, N) = \operatorname{Hom}_{\operatorname{GrMod} A}(M, N)$ the set of all $A$-module homomorphisms $f : M \to N$ preserving degrees, and by $\operatorname{Ext}^i_A(M, N) = \operatorname{Ext}^i_{\operatorname{GrMod} A}(M, N)$ its right derived functors. We further define the graded $k$-vector space

$$\underline{\operatorname{Ext}}^i_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^i_A\bigl(M, N(n)\bigr).$$

Note that if $A$ is right noetherian and $M \in \operatorname{grmod} A$, then it is easy to see that $\underline{\operatorname{Ext}}^i_A(M, N) \cong \operatorname{Ext}^i_{\operatorname{Mod} A}(M, N)$ as vector spaces where we forget graded structures in the left-hand side.

We say that a module $M \in \operatorname{GrMod} A$ is torsion if it is a direct limit of right bounded modules. We denote by $\operatorname{Tors} A$ the full subcategory of $\operatorname{GrMod} A$ consisting of torsion modules, and $\operatorname{Tails} A := \operatorname{GrMod} A / \operatorname{Tors} A$ the quotient category. The natural functor $\pi : \operatorname{GrMod} A \to \operatorname{Tails} A$ is exact and has a right adjoint $\omega : \operatorname{Tails} A \to \operatorname{GrMod} A$. For $M \in \operatorname{GrMod} A$, we often write $\mathcal{M} := \pi M \in \operatorname{Tails} A$. The set of morphisms in $\operatorname{Tails} A$ is denoted by $\operatorname{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = \operatorname{Hom}_{\operatorname{Tails} A}(\mathcal{M}, \mathcal{N})$, and we define

$$\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^i_{\mathcal{A}}\bigl(\mathcal{M}, \mathcal{N}(n)\bigr)$$

as before.

Let $A$ be a connected graded algebra with the unique maximal homogeneous ideal $\mathfrak{m} := A_{\geqslant 1}$. We will define two notions of Cohen–Macaulay. The local cohomology modules of $M \in \operatorname{GrMod} A$ are defined by

$$\underline{\mathrm{H}}^i_{\mathfrak{m}}(M) := \lim_{n \to \infty} \underline{\operatorname{Ext}}^i_A(A/A_{\geqslant n}, M) \in \operatorname{GrMod} A.$$

For $M \in \operatorname{GrMod} A$, we define the following numbers:

$$\begin{aligned} j(M) &:= \inf\bigl\{i \bigm| \underline{\operatorname{Ext}}^i_A(M, A) \neq 0\bigr\}, \\ \operatorname{depth} M &:= \inf\bigl\{i \bigm| \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \neq 0\bigr\}, \\ \operatorname{ldim} M &:= \sup\bigl\{i \bigm| \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \neq 0\bigr\}. \end{aligned}$$

We say that $A$ satisfies Cohen–Macaulay property with respect to GK-dimension if

$$j(M) + \operatorname{GKdim} M = \operatorname{GKdim} A < \infty$$

for all $M \in \operatorname{grmod} A$. We say that $M$ is Cohen–Macaulay if $\operatorname{depth} M = \operatorname{ldim} M < \infty$. We say that $A$ satisfies the condition $\chi$ if $\underline{\mathrm{H}}^i_{\mathfrak{m}}(M)$ are right bounded for all $M \in \operatorname{grmod} A$ and all $i \in \mathbb{N}$.

The following classes of algebras are important in noncommutative projective geometry.

**Definition 1.1.** Let $A$ be a noetherian connected graded algebra.

(1) We say that $A$ is AS Cohen–Macaulay if
- $A$ and $A^o$ satisfy the condition $\chi$, and
- $A$ and $A^o$ are Cohen–Macaulay as graded modules over themselves.

(2) We say that $A$ is AS-Gorenstein if
- $A$ is AS Cohen–Macaulay, and
- $\operatorname{id} A = \operatorname{id} A^o < \infty$.

(3) We say that $A$ is a quantum polynomial algebra if
- $A$ is AS-Gorenstein,
- $\operatorname{gldim} A = \operatorname{gldim} A^o = d < \infty$,

- $H_A(t) = (1-t)^{-d}$, and
- $A$ satisfies Cohen–Macaulay property with respect to GK-dimension.

If $A$ is a noetherian AS Cohen–Macaulay algebra of depth $A = d$, then the graded $A$-$A$ bimodule $\omega_A := \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)^*$ is called the balanced dualizing module [10]. It is known that a noetherian AS Cohen–Macaulay algebra is AS-Gorenstein if and only if there are a graded algebra automorphism $\nu \in \operatorname{Aut} A$ called the generalized Nakayama automorphism, and an integer $\ell \in \mathbb{Z}$ called the Gorenstein parameter, such that $\omega_A \cong A_\nu(-\ell)$ as graded $A$-$A$ bimodules. Note that every artinian connected graded algebra is AS Cohen–Macaulay with the balanced dualizing module $\omega_A = A^*$. It follows that every Frobenius connected graded algebra is AS-Gorenstein and the generalized Nakayama automorphism is the usual Nakayama automorphism in this case (see [15]).

A linear resolution of $M \in \operatorname{grmod} A$ is a free resolution of $M$ of the form

$$\cdots \to A(-3)^{\oplus r_3} \to A(-2)^{\oplus r_2} \to A(-1)^{\oplus r_1} \to A^{\oplus r_0} \to M \to 0$$

for some $r_i \in \mathbb{N}$. The full subcategory of $\operatorname{grmod} A$ consisting of modules having linear resolutions is denoted by $\operatorname{lin} A$. We say that $A$ is Koszul if $k := A/\mathfrak{m} \in \operatorname{lin} A$. It is known that $A$ is Koszul if and only if $A^o$ is Koszul. Every Koszul algebra $A$ is quadratic, that is, $A = T(V)/(R)$ where $R \subset V \otimes_k V$ is a subspace and $(R)$ is the two-sided ideal of $T(V)$ generated by $R$. A useful fact about a Koszul algebra $A$ is that there is a duality $E_A : \operatorname{lin} A \leftrightarrow \operatorname{lin} A^! : E_{A^!}$ where $A^!$ is the quadratic dual of $A$. It is known that $A^!$ is also Koszul, called the Koszul dual of $A$. We refer to [18] for details on Koszul algebras.

For the rest of this paper, we only consider graded algebras finitely generated in degree 1 over a base field $k$. Such an algebra can be written as $A = T(V)/I$ where $V$ is a finite dimensional vector space over $k$, $T(V)$ is the tensor algebra of $V$ over $k$, and $I$ is a homogeneous ideal of $T(V)$. We often fix this presentation of an algebra $A$.

## 2. The condition (EE) over AS-Gorenstein algebras

The purpose of this section is to produce a simple example of an algebra not satisfying (**EE**).

### 2.1. The condition (*PC*)

First, we will define the notion of complete point module and the condition (**PC**).

**Definition 2.1.** Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1. A point module over $A$ is a cyclic module $M \in \operatorname{grmod} A$ such that

$$H_M(t) = \sum_{i=0}^{\infty} t^i = 1 + t + t^2 + t^3 + t^4 + \cdots.$$

The full subcategory of $\operatorname{grmod} A$ consisting of point modules over $A$ is denoted by $\operatorname{pmod} A$. The point module sequence of a point module $M \in \operatorname{pmod} A$ is a sequence of points $\{p_0, p_1, p_2, p_3, \ldots\}$ where $p_n := \mathcal{V}(\{f \in V = A_1 \mid M_n f = 0\}) \subset \mathbb{P}(V^*)$.

A point module $M \in \operatorname{pmod} A$ is called complete if $(\omega\pi M)_{\geqslant n}(n) \in \operatorname{pmod} A$ for all $n \in \mathbb{Z}$. The complete point module sequence of a complete point module $M \in \operatorname{pmod} A$ is a sequence of points $\{\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, \ldots\}$ where $p_n := \mathcal{V}(\{f \in V = A_1 \mid (\omega\pi M)_n f = 0\}) \subset \mathbb{P}(V^*)$. We say that $A$ satisfies (**PC**) if every point module is complete.

Every quantum polynomial algebra satisfies (**PC**) by [15, Proposition 6.6]. (Recall that if $A$ is a quantum polynomial algebra, then every point module over $A$ has a linear resolution, that is, $\operatorname{pmod} A = \operatorname{plin} A := \operatorname{pmod} A \cap \operatorname{lin} A$ [13, Corollary 5.7].) We will find below a larger class of algebras satisfying (**PC**).

Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1. Note that if $M \in \operatorname{pmod} A$, then $M_{\geqslant n}(n) \in \operatorname{pmod} A$ for all $n \in \mathbb{N}$. For a point $p \in \mathbb{P}(V^*)$, we define a module $M_p := A/p^{\perp}A \in \operatorname{grmod} A$ where $p^{\perp} = \{f \in A_1 = V \mid f(p) = 0\}$. By symmetry, we define ${}_pM := A/Ap^{\perp} \in \operatorname{grmod} A^o$. We say that $A$ is quasi-geometric if every point module over $A$ is isomorphic to $M_p$ for some $p \in \mathbb{P}(V^*)$. In this case, we define the geometric pair $\mathcal{P}(A) = (E, \sigma)$ where $E = \{p \in \mathbb{P}(V^*) \mid M_p \in \operatorname{pmod} A\}$ and $\sigma : E \to E$ is the map defined by $(M_p)_{\geqslant 1}(1) \cong M_{\sigma(p)}$ (see [13]).

**Lemma 2.2.** *Let $A$ be a graded algebra, and $M \in \operatorname{GrMod} A$. For $f \in A_i$, if $M_{j-i}f = 0$, then $f(M_j)^* = 0$.*

**Proof.** Let $\phi \in (M_j)^*$ so that $\phi : M_j \to k$ is a linear map. For $f \in A_i$ and $m \in M_{j-i}$, $mf \in M_j$ and $(f\phi)(m) = \phi(mf)$. So if $mf = 0$ for all $m \in M_{j-i}$, then $f\phi = 0$ for all $\phi \in (M_j)^*$. $\square$

**Theorem 2.3.** *Let $A$ be a noetherian AS Cohen–Macaulay algebra of* $\operatorname{depth} A = d$. *If*

$$\underline{\operatorname{Ext}}_A^{d-1}(-, \omega_A)(1) : \operatorname{pmod} A \quad \leftrightarrow \quad \operatorname{pmod} A^o : \underline{\operatorname{Ext}}_{A^o}^{d-1}(-, \omega_A)(1)$$

*is a duality, then $A$ and $A^o$ satisfy (**PC**). In addition, if $A$ and $A^o$ are quasi-geometric, then $\mathcal{P}(A) = (E, \sigma)$ and $\mathcal{P}(A^o) = (E, \sigma^{-1})$ for some bijection $\sigma : E \to E$. Moreover, for a point $p \in E$, the complete point module sequence of $M_p \in \operatorname{pmod} A$ is $\{\sigma^n(p)\}_{n\in\mathbb{Z}}$.*

**Proof.** Since $A$ is a noetherian AS Cohen–Macaulay algebra, for $M \in \operatorname{GrMod} A$ and $i \in \mathbb{Z}$, we have $\underline{\mathrm{H}}_{\mathfrak{m}}^i(M)^* \cong \underline{\operatorname{Ext}}_A^{d-i}(M, \omega_A)$ in $\operatorname{GrMod} A^o$ by local duality [21, Theorem 5.1]. By assumption, if $M \in \operatorname{pmod} A$, then $\underline{\operatorname{Ext}}_A^{d-1}(M, \omega_A)(1) \in \operatorname{pmod} A^o$, so

$$H_{\underline{\mathrm{H}}_{\mathfrak{m}}^1(M)}(t) = H_{\underline{\operatorname{Ext}}_A^{d-1}(M,\omega_A)}\big(t^{-1}\big) = t^{-1}H_{\underline{\operatorname{Ext}}_A^{d-1}(M,\omega_A)(1)}\big(t^{-1}\big) = t^{-1} + t^{-2} + t^{-3} + \cdots.$$

By [1, Proposition 7.2(2)], there is an exact sequence

$$0 \to \underline{\mathrm{H}}_{\mathfrak{m}}^0(M) \cong 0 \to M \to \omega\pi M \to \underline{\mathrm{H}}_{\mathfrak{m}}^1(M) \to 0,$$

so

$$(\omega\pi M)_i \cong \begin{cases} M_i \cong k & \text{if } i \geqslant 0, \\ \underline{\mathrm{H}}_{\mathfrak{m}}^1(M)_i \cong k & \text{if } i < 0. \end{cases}$$

By [1, Proposition 7.2(2)] again, there is an exact sequence

$$0 \to \underline{\mathrm{H}}_{\mathfrak{m}}^0(\omega\pi M) \to \omega\pi M \to \omega\pi(\omega\pi M) \cong \omega\pi M \to \underline{\mathrm{H}}_{\mathfrak{m}}^1(\omega\pi M) \to 0,$$

so $\underline{\mathrm{H}}_{\mathfrak{m}}^0(\omega\pi M) = 0$, hence $\omega\pi M$ is torsion-free. It follows that $(\omega\pi M)_{\geqslant n}$ is cyclic, so $(\omega\pi M)_{\geqslant n}(n) \in \operatorname{pmod} A$ for all $n \in \mathbb{Z}$, hence $A$ satisfies (**PC**). By symmetry, $A^o$ satisfies (**PC**).

If $A$ is quasi-geometric with $\mathcal{P}(A) = (E, \sigma)$, define a map $\tau : E \to E$ by $(\omega\pi M_p)_{\geqslant -1}(-1) = M_{\tau(p)} \in \operatorname{pmod} A$. Since

$$M_{\sigma\tau(p)} \cong (M_{\tau(p)})_{\geqslant 1}(1) = \big((\omega\pi M_p)_{\geqslant -1}(-1)\big)_{\geqslant 1}(1) \cong (\omega\pi M_p)_{\geqslant 0} \cong M_p,$$

$\sigma\tau = \mathrm{id}_E$. Since

$$\begin{aligned} M_{\tau\sigma(p)} &= (\omega\pi M_{\sigma(p)})_{\geqslant -1}(-1) \cong \big(\omega\pi\big((M_p)_{\geqslant 1}(1)\big)\big)_{\geqslant -1}(-1) \\ &\cong (\omega\pi M_p)(1)_{\geqslant -1}(-1) \cong (\omega\pi M_p)_{\geqslant 0} \cong M_p, \end{aligned}$$

$\tau\sigma = \mathrm{id}_E$, so $\sigma : E \to E$ is a bijection and $\tau = \sigma^{-1} : E \to E$. It follows that, for a point $p \in E$, the complete point module sequence of $M_p \in \operatorname{pmod} A$ is $\{\sigma^n(p)\}_{n\in\mathbb{Z}}$.

By symmetry, if $A^o$ is quasi-geometric with $\mathcal{P}(A^o) = (E^o, \sigma^o)$, then $\sigma^o : E^o \to E^o$ is a bijection. Define a map $\phi : E \to E^o$ by $\underline{\mathrm{Ext}}_A^{d-1}(M_p, \omega_A)(1) \cong {}_{\phi(p)}M \in \operatorname{pmod} A^o$. For $M = M_p \in \operatorname{pmod} A$,

$$\begin{aligned}\phi(p) &= \mathcal{V}\big(\big\{f \in V = A_1 \bigm| f\, \underline{\mathrm{Ext}}_A^{d-1}(M_p, \omega_A)(1)_0 = 0\big\}\big)\\ &= \mathcal{V}\big(\big\{f \in V = A_1 \bigm| f\, \underline{\mathrm{Ext}}_A^{d-1}(M_p, \omega_A)_1 = 0\big\}\big)\\ &= \mathcal{V}\big(\big\{f \in V = A_1 \bigm| f\big(\underline{\mathrm{H}}_{\mathfrak{m}}^1(M)_{-1}\big)^* = 0\big\}\big)\\ &\subset \mathcal{V}\big(\big\{f \in V = A_1 \bigm| \underline{\mathrm{H}}_{\mathfrak{m}}^1(M)_{-2} f = 0\big\}\big)\\ &= \mathcal{V}\big(\big\{f \in V = A_1 \bigm| (\omega\pi M)_{-2} f = 0\big\}\big)\\ &= \sigma^{-2}(p)\end{aligned}$$

by Lemma 2.2, so $E = E^o$ and $\phi = \sigma^{-2} : E \to E$. A short exact sequence

$$0 \to M_{\geqslant 1} \to M \to M/M_{\geqslant 1} \to 0$$

induces an exact sequence

$$\underline{\mathrm{H}}_{\mathfrak{m}}^1(M_{\geqslant 1}) \to \underline{\mathrm{H}}_{\mathfrak{m}}^1(M) \to \underline{\mathrm{H}}_{\mathfrak{m}}^1(M/M_{\geqslant 1}) \cong 0,$$

so there is an injection of shifted point modules

$$\begin{aligned}{}_{\sigma^{-2}(p)}M(-1) &= {}_{\phi(p)}M(-1) = \underline{\mathrm{Ext}}_A^{d-1}(M, \omega_A) \cong \underline{\mathrm{H}}_{\mathfrak{m}}^1(M)^*\\ &\to \underline{\mathrm{H}}_{\mathfrak{m}}^1(M_{\geqslant 1})^* \cong \underline{\mathrm{Ext}}_A^{d-1}(M_{\geqslant 1}, \omega_A) \cong \underline{\mathrm{Ext}}_A^{d-1}\big(M_{\geqslant 1}(1), \omega_A\big)(1)\\ &\cong \underline{\mathrm{Ext}}_A^{d-1}(M_{\sigma(p)}, \omega_A)(1) = {}_{\phi\sigma(p)}M = {}_{\sigma^{-1}(p)}M.\end{aligned}$$

It follows that

$${}_{\sigma^o\sigma^{-1}(p)}M = {}_{\sigma^{-1}(p)}M_{\geqslant 1}(1) \cong {}_{\sigma^{-2}(p)}M(-1)_{\geqslant 1}(1) \cong {}_{\sigma^{-2}(p)}M$$

for all $p \in E$, so $\sigma^o = \sigma^{-1} : E \to E$. □

The following corollary extends [15, Theorem 6.3].

**Corollary 2.4.** *If $A$ is an AS Cohen–Macaulay Koszul algebra which is a graded quotient algebra of a quantum polynomial algebra, or an FBN (fully bounded noetherian) AS-Gorenstein Koszul algebra, then $A$ and $A^o$ satisfy (**PC**), and $\mathcal{P}(A) = (E, \sigma)$ and $\mathcal{P}(A^o) = (E, \sigma^{-1})$ for some bijection $\sigma : E \to E$. In particular, there is a bijection between isomorphism classes of point modules over $A$ and those over $A^o$.*

**Proof.** If $A$ is an AS Cohen–Macaulay Koszul algebra which is a graded quotient algebra of a quantum polynomial algebra, or an FBN AS-Gorenstein Koszul algebra, then $A$ and $A^o$ are quasi-geometric by [13, Corollary 5.7, Theorems 5.8 and 3.8], and $\underline{\mathrm{Ext}}_A^{d-1}(-, \omega_A)(1) : \operatorname{pmod} A \leftrightarrow \operatorname{pmod} A^o : \underline{\mathrm{Ext}}_{A^o}^{d-1}(-, \omega_A)(1)$ is a duality by [15, Lemma 4.6], so the result follows from Theorem 2.3. □

*2.2. The condition (**EE**)*

In this subsection, we will give a geometric way of constructing algebras not satisfying (**EE**). The basic idea is to reduce the problem in the category $\operatorname{GrMod} A$ to that in the category $\operatorname{Tails} A$. So we first compare Ext-groups in $\operatorname{GrMod} A$ with those in $\operatorname{Tails} A$.

**Lemma 2.5.** *Let $A$ be a right noetherian locally finite connected graded algebra satisfying $\chi$, and $M, N \in$ $\operatorname{grmod} A$. Then, for each $i \in \mathbb{N}$, $\dim_k \underline{\operatorname{Ext}}^i_A(M, N) < \infty$ if and only if $\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$ is right bounded.*

**Proof.** For each $i \in \mathbb{N}$,

(1) $\dim_k \underline{\operatorname{Ext}}^i_A(M, N)_n < \infty$ for all $n \in \mathbb{Z}$ by [1, Proposition 3.1(3)],
(2) $\underline{\operatorname{Ext}}^i_A(M, N)_n = 0$ for all $n \ll 0$ by [1, Proposition 3.1(1)(c)], and
(3) $\underline{\operatorname{Ext}}^i_A(M, N)_{\geqslant n} \cong \underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_{\geqslant n}$ as graded vector spaces for all $n \gg 0$ by [1, Corollary 7.3(2)],

hence the result. $\square$

We will see below that Ext-groups in $\operatorname{Tails} A$ are also controlled by geometry. For an abelian category $\mathcal{C}$, we denote by $\mathcal{D}(\mathcal{C})$ ($\mathcal{D}^b(\mathcal{C})$) the (bounded) derived category of $\mathcal{C}$. For $X \in \mathcal{D}(\mathcal{C})$ and $n \in \mathbb{Z}$, $X[n] \in \mathcal{D}(\mathcal{C})$ is defined by $(X[n])^i := X^{n+i}$ for $i \in \mathbb{Z}$. The following lemma is well known. We will include the proof for the convenience of the reader.

**Lemma 2.6.** *Let $A$ be a right noetherian connected graded algebra. For $K \in \operatorname{GrMod} A$, we have an isomorphism of functors*

$$\mathrm{R}\underline{\operatorname{Hom}}_{\mathcal{A}}(\mathcal{K}, -) \cong \mathrm{R}\underline{\operatorname{Hom}}_A\big(K, \mathrm{R}\omega(-)\big) : \mathcal{D}^b(\operatorname{Tails} A) \to \mathcal{D}(\operatorname{GrMod} k).$$

**Proof.** Note that $\operatorname{GrMod} A, \operatorname{Tails} A, \operatorname{GrMod} k$ are all abelian categories having enough injectives. Since $\underline{\operatorname{Hom}}_A(K, -) : \operatorname{GrMod} A \to \operatorname{GrMod} k$ is a left exact functor, and $\omega : \operatorname{Tails} A \to \operatorname{GrMod} A$ is a left exact functor preserving injectives by [1, Proposition 7.1(1)],

$$\begin{aligned}\mathrm{R}\underline{\operatorname{Hom}}_{\mathcal{A}}(\mathcal{K}, -) &\cong \mathrm{R}\big(\underline{\operatorname{Hom}}_A\big(K, \omega(-)\big)\big) = \mathrm{R}\big(\underline{\operatorname{Hom}}_A(K, -) \circ \omega(-)\big) \\ &\cong \mathrm{R}\underline{\operatorname{Hom}}_A(K, -) \circ \mathrm{R}\omega(-) = \mathrm{R}\underline{\operatorname{Hom}}_A\big(K, \mathrm{R}\omega(-)\big)\end{aligned}$$

as functors $\mathcal{D}^b(\operatorname{Tails} A) \to \mathcal{D}(\operatorname{GrMod} k)$ by [19, Corollary 10.8.3]. $\square$

**Definition 2.7.** Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1. A cyclic graded module of the form $K = A/fA \in \operatorname{grmod} A$ where $f \in A_1 = V$ is a homogeneous regular element of degree 1 is called a hyperplane module over $A$ (associated to the hyperplane $H = \mathcal{V}(f) \subset \mathbb{P}(V^*)$).

**Proposition 2.8.** *Let $A = T(V)/I$ be a noetherian AS-Gorenstein algebra satisfying Cohen–Macaulay property with respect to GK-dimension. If $M \in \operatorname{pmod} A$ is a complete point module with the complete point module sequence $\{p_n\}_{n\in\mathbb{Z}}$, and $K = A/fA \in \operatorname{GrMod} A$ is a hyperplane module where $f \in A_1 = V$ is a homogeneous regular element of degree 1, then*

$$\dim_k \underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{K}, \mathcal{M})_n = \begin{cases} k & \text{if } i = 0, 1 \text{ and } p_n \in \mathcal{V}(f), \\ 0 & \text{otherwise.}\end{cases}$$

**Proof.** Since $A$ is a noetherian AS-Gorenstein algebra satisfying Cohen–Macaulay property with respect to GK-dimension, $\operatorname{GKdim} A = j(k) + \operatorname{GKdim} k = \operatorname{depth} A$, so $\operatorname{depth} M \leqslant \operatorname{ldim} M = \operatorname{GKdim} M = 1$

by [12, Remark 2.4]. By [1, Proposition 7.2(2)], $h^i(\mathrm{R}\omega\mathcal{M}) \cong \underline{\mathrm{H}}^{i+1}_{\mathfrak{m}}(M) = 0$ for all $i \geqslant 1$, so $\mathrm{R}\omega\mathcal{M} \cong h^0(\mathrm{R}\omega\mathcal{M}) \cong \omega\mathcal{M}$ in $\mathcal{D}(\mathrm{GrMod}\, A)$. By Lemma 2.6, $\mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{K}, \mathcal{M}) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(K, \mathrm{R}\omega\mathcal{M}) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(K, \omega\mathcal{M})$. Consider the long exact sequence

$$\begin{aligned} \cdots \to \underline{\mathrm{Ext}}^{i-1}_A\big(K, \omega\mathcal{M}/(\omega\mathcal{M})_{\geqslant n}\big) &\to \underline{\mathrm{Ext}}^i_A\big(K, (\omega\mathcal{M})_{\geqslant n}\big) \\ \to \underline{\mathrm{Ext}}^i_A(K, \omega\mathcal{M}) &\to \underline{\mathrm{Ext}}^i_A\big(K, \omega\mathcal{M}/(\omega\mathcal{M})_{\geqslant n}\big) \to \cdots. \end{aligned}$$

Since $\underline{\mathrm{Ext}}^i_A(K, \omega\mathcal{M}/(\omega\mathcal{M})_{\geqslant n})_{\geqslant n} = 0$ for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$ by [1, Proposition 3.1(2)], $\underline{\mathrm{Ext}}^i_A(K, (\omega\mathcal{M})_{\geqslant n})_{\geqslant n} \cong \underline{\mathrm{Ext}}^i_A(K, \omega\mathcal{M})_{\geqslant n}$, so

$$\underline{\mathrm{Ext}}^i_{\mathcal{A}}(\mathcal{K}, \mathcal{M})_n \cong \underline{\mathrm{Ext}}^i_A(K, \omega\mathcal{M})_n \cong \underline{\mathrm{Ext}}^i_A\big(K, (\omega\mathcal{M})_{\geqslant n}\big)_n \cong \underline{\mathrm{Ext}}^i_A\big(K, (\omega\mathcal{M})_{\geqslant n}(n)\big)_0$$

for all $i \in \mathbb{N}$ and all $n \in \mathbb{Z}$. Since $(\omega\mathcal{M})_{\geqslant n}(n)$ is a point module with the point module sequence $\{p_n, p_{n+1}, p_{n+2}, p_{n+3}, \ldots\}$, the result follows from the proof of [13, Theorem 6.1]. □

Next, we will see below that an asymmetric behavior of Ext in the category $\mathrm{Tails}\, A$ is controlled by the generalized Nakayama automorphism.

**Definition 2.9.** We say that an abelian category $\mathcal{C}$ has finite global dimension if there is an integer $n \in \mathbb{N}$ such that $\mathrm{Ext}^i_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) = 0$ for all $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ and all $i > n$.

**Example 2.10.** The following are examples of graded algebras $A$ such that $\mathrm{Tails}\, A$ has finite global dimension.

(1) If $A$ is a commutative graded algebra finitely generated in degree 1 and $X = \mathrm{Proj}\, A$ is the associated projective scheme, then $\mathrm{Tails}\, A \cong \mathrm{Mod}\, X$ where $\mathrm{Mod}\, X$ is the category of quasi-coherent sheaves on $X$, so $\mathrm{Tails}\, A$ has finite global dimension if and only if $\mathrm{Proj}\, A$ is smooth. So we may think of the condition "$\mathrm{Tails}\, A$ has finite global dimension" as the condition "$\mathrm{Proj}\, A$ is smooth" even if $A$ is not commutative.
(2) If $A$ has finite global dimension, then $\mathrm{Tails}\, A$ has finite global dimension.
(3) If $A$ is a right noetherian graded algebra generated in degree 1, then $\mathrm{Tails}\, A^{(r)} \cong \mathrm{Tails}\, A$ for any $r \in \mathbb{N}^+$ by [1, Proposition 5.10(3)] where $A^{(r)}$ is the $r$th Veronese subalgebra of $A$, so if $A$ has finite global dimension, then $\mathrm{Tails}\, A^{(r)}$ has finite global dimension.
(4) If $A$ is an FBN AS Cohen–Macaulay algebra of finite CM-representation type, then, for any $M, N \in \mathrm{grmod}\, A$, $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) = 0$ for all $i \geqslant \mathrm{depth}\, A$ by [7, Theorem 2.5], so $\mathrm{tails}\, A$ has finite global dimension.

Let $A$ be a graded algebra and $\tau \in \mathrm{Aut}\, A$ a graded algebra automorphism. For a graded module $M \in \mathrm{GrMod}\, A$, we define a new graded module $M_\tau \in \mathrm{GrMod}\, A$ by $M_\tau = M$ as a graded vector space with the new right action $m * a := m\tau(a)$ for $m \in M, a \in A$.

**Lemma 2.11.** *Let $A$ be a noetherian AS-Gorenstein algebra of injective dimension $d$ and Gorenstein parameter $\ell$, and let $\nu \in \mathrm{Aut}\, A$ be the generalized Nakayama automorphism of $A$. If $\mathrm{Tails}\, A$ has finite global dimension, then, for $M, N \in \mathrm{grmod}\, A$ and all $i \in \mathbb{Z}$,*

$$\underline{\mathrm{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \cong \underline{\mathrm{Ext}}^{d-1-i}_{\mathcal{A}}(\mathcal{N}, \mathcal{M}_\nu)^*(\ell)$$

*as graded vector spaces. In particular, for all $n \in \mathbb{Z}$,*

$$\underline{\mathrm{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_n \cong \underline{\mathrm{Ext}}^{d-1-i}_{\mathcal{A}}(\mathcal{N}, \mathcal{M}_\nu)_{-n-\ell}$$

*as vector spaces.*

**Proof.** Since $A$ is a noetherian AS-Gorenstein algebra, $A_\nu(-\ell)[d]$ is the balanced dualizing complex for $A$ in the sense of [20]. Since $\operatorname{Tails} A$ has finite global dimension,

$$- \otimes^L_{\mathcal{A}} \mathcal{A}_\nu(-\ell)[d-1] : \mathcal{D}^b(\operatorname{tails} A) \to \mathcal{D}^b(\operatorname{tails} A)$$

is the Serre functor by [17, Theorem A.4], so, for $M, N \in \operatorname{grmod} A$,

$$\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \cong \underline{\operatorname{Ext}}^{-i}_{\mathcal{A}}\big(\mathcal{N}, \mathcal{M} \otimes^L_{\mathcal{A}} \mathcal{A}_\nu(-\ell)[d-1]\big)^* \cong \underline{\operatorname{Ext}}^{d-1-i}_{\mathcal{A}}(\mathcal{N}, \mathcal{M}_\nu)^*(\ell)$$

as graded vector spaces. $\square$

Let $A = T(V)/I$ be a graded algebra. If $\tau \in \operatorname{Aut} A$ is a graded algebra automorphism, then it restricts to an automorphism $\tau : V \to V$, so its dual induces an automorphism $\tau^* : \mathbb{P}(V^*) \to \mathbb{P}(V^*)$. If $A$ is quasi-geometric with $\mathcal{P}(A) = (E, \sigma)$, then it is easy to see that $\tau^*$ restricts to a bijection $\tau^* : E \to E$ (see [15]).

**Theorem 2.12.** *Let $A = T(V)/I$ be a quasi-geometric FBN AS-Gorenstein algebra satisfying (**PC**) with $\mathcal{P}(A) = (E, \sigma)$ such that* Tails $A$ *has finite global dimension, and let $\nu \in$* Aut $A$ *be the generalized Nakayama automorphism of $A$. If there are a point $p \in E$ and a hyperplane $\mathcal{V}(f) \subset \mathbb{P}(V^*)$ where $f \in A_1 = V$ is a homogeneous regular element of degree* 1 *such that $\#\{n \in \mathbb{N} \mid \sigma^n(p) \in \mathcal{V}(f)\} = \infty$ but $\#\{n \in \mathbb{N} \mid \sigma^{-n}\nu^*(p) \in \mathcal{V}(f)\} < \infty$, then $A$ does not satisfy (**EE**).*

**Proof.** By [22, Theorem 3.1(1)], $A$ satisfies Cohen–Macaulay property with respect to GK-dimension. Let $M = M_p$ be a point module and $K = A/fA$ a hyperplane module. Since $\{\sigma^n(p)\}_{n \in \mathbb{Z}}$ is a complete point module sequence of $M$ by Theorem 2.3, if $\#\{n \in \mathbb{N} \mid \sigma^n(p) \in \mathcal{V}(f)\} = \infty$, then, for $i = 0, 1$, $\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{K}, \mathcal{M})_n \neq 0$ for infinitely many $n \in \mathbb{N}$ by Proposition 2.8, so $\dim_k \underline{\operatorname{Ext}}^i_A(K, M) = \infty$ by Lemma 2.5. On the other hand, since $\{\sigma^n \nu^*(p)\}_{n \in \mathbb{Z}}$ is a complete point module sequence of $M_\nu$ by [15, Lemma 3.2(1)] and Theorem 2.3, if $\#\{n \in \mathbb{N} \mid \sigma^{-n}\nu^*(p) \in \mathcal{V}(f)\} < \infty$, then $\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{K}, \mathcal{M}_\nu)_{-n} = 0$ for all $i \in \mathbb{N}$ and all $n \gg 0$ by Proposition 2.8. It follows that $\underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{K})$ are right bounded for all $i \in \mathbb{N}$ by Lemma 2.11, hence $\dim_k \underline{\operatorname{Ext}}^i_A(M, K) < \infty$ for all $i \in \mathbb{N}$ by Lemma 2.5. $\square$

**Remark 2.13.** In the above arguments, techniques of noncommutative projective geometry play an essential role in the following sense: since $\mathcal{D}^b(\operatorname{grmod} A)$ almost never has Serre functor, an asymmetric behavior of Ext in the category $\operatorname{GrMod} A$ is not controlled by the generalized Nakayama automorphism. In fact, although we know that $\underline{\operatorname{Ext}}^i_A(M, N)_{\geqslant n} \cong \underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_{\geqslant n}$ and $\underline{\operatorname{Ext}}^i_A(N, M_\nu)_{\geqslant n} \cong \underline{\operatorname{Ext}}^i_{\mathcal{A}}(\mathcal{N}, \mathcal{M}_\nu)_{\geqslant n}$ for all $n \gg 0$ by [1, Corollary 7.3(2)], $\underline{\operatorname{Ext}}^\bullet_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_{\gg 0}$ and $\underline{\operatorname{Ext}}^\bullet_{\mathcal{A}}(\mathcal{N}, \mathcal{M}_\nu)_{\gg 0}$ are not related in general, so $\underline{\operatorname{Ext}}^\bullet_A(M, N)_{\gg 0}$ and $\underline{\operatorname{Ext}}^\bullet_A(N, M_\nu)_{\gg 0}$ are not related at all. What we know is that $\underline{\operatorname{Ext}}^\bullet_{\mathcal{A}}(\mathcal{M}, \mathcal{N})_{\gg 0}$ and $\underline{\operatorname{Ext}}^\bullet_{\mathcal{A}}(\mathcal{N}, \mathcal{M}_\nu)_{\ll 0}$ are related by Lemma 2.11. (We would like to emphasize this point, because some of the readers might think that the above result is trivial by a simple application of the Serre functor.) That is why we must pass through the category Tails $A$ and use rather special modules, namely, complete point modules in the above arguments.

**Corollary 2.14.** *Let $A = T(V)/I$ be a quasi-geometric FBN AS-Gorenstein domain satisfying (**PC**) with $\mathcal{P}(A) = (E, \sigma)$ such that* Tails $A$ *has finite global dimension, and let $\nu \in$* Aut $A$ *be the generalized Nakayama automorphism of $A$. If*

- *$k$ is an infinite field,*
- $\dim_k V \geqslant 3$,
- *$|\sigma| < \infty$, and*
- *there is a point $p \in E$ such that $\sigma^i(p) \neq \nu^*(p)$ for any $i \in \mathbb{Z}$,*

*then $A$ does not satisfy (**EE**).*

**Proof.** Since $A$ is a domain, any homogeneous element $f \in A_1$ of degree 1 is regular. Since $k$ is infinite, $\dim_k V \geqslant 3$, and the orbit $S$ of $\nu^*(p)$ under $\sigma$ is a finite set not containing $p$, there is a hyperplane $\mathcal{V}(f) \subset \mathbb{P}(V^*)$ such that $p \in \mathcal{V}(f)$ but $\mathcal{V}(f) \cap S = \emptyset$. Since $|\sigma| < \infty$, $\sigma^n(p) \in \mathcal{V}(f)$ for infinitely many $n \in \mathbb{N}$ but $\sigma^{-n}\nu^*(p) \notin \mathcal{V}(f)$ for any $n \in \mathbb{N}$, hence the result follows from Theorem 2.12. $\square$

**Example 2.15.** Let $A = k\langle x, y, z\rangle/(zy - \alpha yz, xz - \beta zx, yx - \gamma xy)$ where $\alpha, \beta, \gamma \in k$ such that $\alpha\beta\gamma \neq 0, 1$. It is well known that $\mathcal{P}(A) = (E, \sigma)$ where $E = \mathcal{V}(xyz) \subset \mathbb{P}(V^*)$ and

$$\begin{aligned}\sigma(0, b, c) &= (0, \alpha b, c),\\ \sigma(a, 0, c) &= (a, 0, \beta c),\\ \sigma(a, b, 0) &= (\gamma a, b, 0).\end{aligned}$$

Since

$$\begin{aligned}(xyz)x &= (\gamma/\beta)x(xyz),\\ (xyz)y &= (\alpha/\gamma)y(xyz),\\ (xyz)z &= (\beta/\alpha)x(xyz)\end{aligned}$$

in $A$, the bijection $\nu^* : E \to E$ induced by the generalized Nakayama automorphism $\nu \in \operatorname{Aut} A$ of $A$ is given by

$$\nu^*(a, b, c) = \big((\gamma/\beta)a, (\alpha/\gamma)b, (\beta/\alpha)c\big)$$

by [20, Theorem 7.18]. Suppose that $k$ is an infinite field, and $\alpha, \beta, \gamma$ are roots of unity so that $|\sigma| < \infty$. If $\alpha^i \neq \beta\gamma$ for any $i \in \mathbb{Z}$, then $\sigma^i(0, 1, 1) = (0, \alpha^i, 1) \neq (0, \alpha/\gamma, \beta/\alpha) = \nu^*(0, 1, 1)$ for any $i \in \mathbb{Z}$, so $A$ does not satisfy (**EE**) by Corollary 2.14. By symmetry, if $\beta^i \neq \gamma\alpha$ for any $i \in \mathbb{Z}$, or $\gamma^i \neq \alpha\beta$ for any $i \in \mathbb{Z}$, then $A$ does not satisfy (**EE**).

## 3. The condition (ee) over Frobenius Koszul algebras

The purpose of this section is to produce a simple example of an algebra not satisfying (**ee**). We will see that there are many analogies between the arguments in this section and the previous one.

### 3.1. The condition (**pc**)

First, we will define the notion of complete co-point module and the condition (**pc**).

**Definition 3.1.** Let $A$ be a noetherian graded algebra. A graded module $N \in \operatorname{grmod} A$ is called totally reflexive if $\underline{\operatorname{Ext}}^i_A(N, A) = 0 = \underline{\operatorname{Ext}}^i_{A^o}(\underline{\operatorname{Hom}}_A(N, A), A)$ for all $i \geqslant 1$, and there is a canonical isomorphism $N \cong \underline{\operatorname{Hom}}_{A^o}(\underline{\operatorname{Hom}}_A(N, A), A)$ in $\operatorname{grmod} A$.

In other words, $N \in \operatorname{grmod} A$ is totally reflexive if and only if $\mathrm{R}\underline{\operatorname{Hom}}_A(N, A) \cong \underline{\operatorname{Hom}}_A(N, A)$ in $\mathcal{D}^b(\operatorname{grmod} A^o)$ and $\mathrm{R}\underline{\operatorname{Hom}}_{A^o}(\mathrm{R}\underline{\operatorname{Hom}}_A(N, A), A) \cong N$ in $\mathcal{D}^b(\operatorname{grmod} A)$. Note that if $A$ is Frobenius, then every finitely generated graded module is totally reflexive.

**Definition 3.2.** Let $A$ be a noetherian graded algebra. An acyclic complex of finitely generated free modules $T$ is called totally acyclic if $\underline{\operatorname{Hom}}_A(T, A)$ is also acyclic.

For a complex of free modules

$$T : \cdots \xrightarrow{\partial^{n-2}} T^{n-1} \xrightarrow{\partial^{n-1}} T^n \xrightarrow{\partial^n} T^{n+1} \xrightarrow{\partial^{n+1}} \cdots,$$

we define $\Omega^n T := \operatorname{Coker} \partial^{n-1}$. For $N \in \operatorname{GrMod} A$, we define $\Omega^n N := \Omega^n F$ where $F$ is the minimal free resolution of $N$.

**Lemma 3.3.** *(See [3, Theorem 3.1].) Let $A$ be a noetherian graded algebra. A graded module $N \in \operatorname{grmod} A$ is totally reflexive if and only if there is a totally acyclic complex $T$ such that $\Omega^0 T \cong N$.*

We call $T$ in the above lemma a complete resolution of $N$. A complete resolution can be defined for more general modules (see [3]).

**Definition 3.4.** Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1. A co-point module over $A$ having a linear resolution is a module $N \in \operatorname{grmod} A$ having a linear resolution of the form

$$\cdots \xrightarrow{v_3\cdot} A(-3) \xrightarrow{v_2\cdot} A(-2) \xrightarrow{v_1\cdot} A(-1) \xrightarrow{v_0\cdot} A \to N \to 0$$

where $v_i \in A_1 = V$. The full subcategory of $\operatorname{grmod} A$ consisting of co-point modules over $A$ having linear resolutions is denoted by $\operatorname{clin} A$.

A co-point module $N \in \operatorname{clin} A$ having a linear resolution is called complete if $N$ has a complete resolution of the form

$$T : \cdots \xrightarrow{v_2\cdot} A(-2) \xrightarrow{v_1\cdot} A(-1) \xrightarrow{v_0} A \xrightarrow{v_{-1}\cdot} A(1) \xrightarrow{v_{-2}\cdot} A(2) \xrightarrow{v_{-3}\cdot} \cdots$$

where $v_i \in A_1 = V$. The complete co-point module sequence of a complete co-point module $N$ is a sequence of points $\{\dots, p_{-2}, p_{-1}, p_0, p_1, p_2, \dots\}$ where $p_i = [v_i] \in \mathbb{P}(V)$ are the images of $v_i \in V$. We say that $A$ satisfies (**pc**) if every co-point module having a linear resolution is complete.

If $A$ is a Frobenius Koszul algebra such that $A^!$ is a quantum polynomial algebra, then $A$ satisfies (**pc**) by the proof of [15, Theorem 6.1(1)]. We will find below a larger class of algebras satisfying (**pc**).

Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1. Note that if $N \in \operatorname{clin} A$, then $\Omega^n N(n) \in \operatorname{clin} A$ for all $n \in \mathbb{N}$. For a point $p \in \mathbb{P}(V)$, we define a module $N_p := A/vA \in \operatorname{grmod} A$ where $v \in A_1 = V$ such that $p = [v] \in \mathbb{P}(V)$. By definition, every co-point module over $A$ having a linear resolution is isomorphic to $N_p$ for some $p \in \mathbb{P}(V)$. We define the co-geometric pair $\mathcal{P}^!(A) = (E, \sigma)$ where $E = \{p \in \mathbb{P}(V) \mid N_p \in \operatorname{clin} A\}$ and $\sigma : E \to E$ is the map defined by $\Omega N_p(1) \cong N_{\sigma(p)}$ (see [13]).

We now prepare two easy lemmas.

**Lemma 3.5.** *Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1, and $\tau \in \operatorname{Aut} A$ a graded algebra automorphism. If $N \in \operatorname{clin} A$ is a complete co-point module with the complete co-point module sequence $\{\dots, p_{-2}, p_{-1}, p_0, p_1, p_2, \dots\}$, then $N_\tau \in \operatorname{clin} A$ is also a complete co-point module with the complete co-point module sequence $\{\dots, \tau^{-1}(p_{-2}), \tau^{-1}(p_{-1}), \tau^{-1}(p_0), \tau^{-1}(p_1), \tau^{-1}(p_2), \dots\}$.*

**Proof.** If

$$\cdots \xrightarrow{v_2\cdot} A(-2) \xrightarrow{v_1\cdot} A(-1) \xrightarrow{v_0\cdot} A \xrightarrow{v_{-1}\cdot} A(1) \xrightarrow{v_{-2}\cdot} \cdots$$

is a complete resolution of $N$ where $p_i = [v_i] \in \mathbb{P}(V)$, then

$$\cdots \xrightarrow{\tau^{-1}(v_2)\cdot} A(-2) \xrightarrow{\tau^{-1}(v_1)\cdot} A(-1) \xrightarrow{\tau^{-1}(v_0)\cdot} A \xrightarrow{\tau^{-1}(v_{-1})\cdot} A(1) \xrightarrow{\tau^{-1}(v_{-2})\cdot} \cdots$$

is a complete resolution of $N_\tau$ where $\tau^{-1}(p_i) = [\tau^{-1}(v_i)] \in \mathbb{P}(V)$. $\square$

**Lemma 3.6.** *Let $A$ be a graded algebra. If $\operatorname{id} A < \infty$, then every acyclic complex of finitely generated free modules is totally acyclic.*

**Proof.** Let $d = \operatorname{id} A < \infty$ and $T$ an acyclic complex of finitely generated free modules. Since $T^{\leqslant d+1-n}[d+1-1]$ is a free resolution of $\Omega^{d+1-n}T$,

$$\begin{aligned}\underline{\mathrm{Ext}}_A^n(T, A) &\cong \underline{\mathrm{Ext}}_A^n\big(T^{\leqslant d+1-n}, A\big) \cong \underline{\mathrm{Ext}}_A^{d+1}\big(T^{\leqslant d+1-n}[d+1-n], A\big)\\ &\cong \underline{\mathrm{Ext}}_A^{d+1}\big(\Omega^{d+1-n}T, A\big) = 0\end{aligned}$$

for all $n \in \mathbb{Z}$, hence the result. $\square$

**Theorem 3.7.** *Let $A$ be a noetherian AS-Gorenstein algebra. If $\mathcal{P}^!(A) = (E, \sigma)$ for some bijection $\sigma : E \to E$, then, for each $p \in E$, $N_p \in \operatorname{clin} A$ is a complete co-point module with the complete co-point module sequence $\{\sigma^n(p)\}_{n\in\mathbb{Z}}$. In particular, $A$ satisfies (**pc**).*

**Proof.** Since $\sigma : E \to E$ is a bijection, for every point $p \in E$, $\sigma^n(p) \in E$ for all $n \in \mathbb{Z}$, so

$$T : \cdots \xrightarrow{\sigma^2(p)\cdot} A(-2) \xrightarrow{\sigma(p)\cdot} A(-1) \xrightarrow{p\cdot} A \xrightarrow{\sigma^{-1}(p)\cdot} A(1) \xrightarrow{\sigma^{-2}(p)\cdot} A(2) \xrightarrow{\sigma^{-3}(p)\cdot} \cdots$$

is an acyclic complex of finitely generated free modules such that $\Omega^0 T \cong N_p$. Since $T$ is totally acyclic by Lemma 3.6, it is a complete resolution of $N_p \in \operatorname{clin} A$. $\square$

We know so far no example of an algebra such that $\sigma : E \to E$ is not a bijection, so we hope that a large class of noetherian AS-Gorenstein algebras satisfy (**pc**). For example, the following corollary provides a class of algebras satisfying (**pc**).

**Corollary 3.8.** *Let $A$ be a noetherian AS-Gorenstein Koszul algebra. If its Koszul dual $A^!$ is a graded quotient algebra of a quantum polynomial algebra, or an FBN AS-Gorenstein Koszul algebra, then $A$ and $A^o$ satisfy (**pc**).*

**Proof.** By [13, Corollary 5.7, Theorems 5.8 and 3.8] and Corollary 2.4, $\mathcal{P}^!(A) = \mathcal{P}(A^!) = (E, \sigma)$ for some bijection $\sigma : E \to E$, so $A$ satisfies (**pc**) by Theorem 3.7. By symmetry, $A^o$ satisfies (**pc**). $\square$

The following are characterizations of algebras satisfying (**pc**).

**Theorem 3.9.** *Let $A$ be a noetherian AS-Gorenstein algebra. Then the following are equivalent*:

(1) *Every co-point module over $A$ and $A^o$ having linear resolution is totally reflexive and*

$$\underline{\mathrm{Hom}}_A(-, A)(1) : \operatorname{clin} A \quad \leftrightarrow \quad \operatorname{clin} A^o : \underline{\mathrm{Hom}}_{A^o}(-, A)(1)$$

*is a duality.*

(2) $\mathcal{P}^!(A) = (E, \sigma)$ *and* $\mathcal{P}^!(A^o) = (E, \sigma^{-1})$ *for some bijection* $\sigma : E \to E$.

(3) *$A$ and $A^o$ satisfy (**pc**).*

**Proof.** (1) $\Rightarrow$ (2): This follows from the proof of [15, Theorem 6.1(1)].

(2) $\Rightarrow$ (3): This follows from Theorem 3.7.

(3) $\Rightarrow$ (1): Suppose that every co-point module $N_{p_0} \in \operatorname{clin} A$ has a complete resolution of the form

$$\cdots \xrightarrow{v_2\cdot} A(-2) \xrightarrow{v_1\cdot} A(-1) \xrightarrow{v_0\cdot} A \xrightarrow{v_{-1}\cdot} A(1) \xrightarrow{v_{-2}\cdot} A(2) \xrightarrow{v_{-3}\cdot} \cdots.$$

Applying the functor $\underline{\mathrm{Hom}}_A(-, A)$, we have an acyclic complex

$$\cdots \xleftarrow{\cdot v_2} A(2) \xleftarrow{\cdot v_1} A(1) \xleftarrow{\cdot v_0} A \xleftarrow{\cdot v_{-1}} A(-1) \xleftarrow{\cdot v_{-2}} A(-2) \xleftarrow{\cdot v_{-3}} \cdots,$$

so it is a complete resolution of $\underline{\mathrm{Hom}}_A(N_{p_0}, A)$. It follows that $\underline{\mathrm{Hom}}_A(N_{p_0}, A)(1) \cong {}_{p_{-2}}N \in \mathrm{clin}\, A^o$, so there is a functor $\underline{\mathrm{Hom}}_A(-, A)(1) : \mathrm{clin}\, A \to \mathrm{clin}\, A^o$. Since $N_{p_0}$ is totally reflexive by Lemma 3.3,

$$\underline{\mathrm{Hom}}_{A^o}\big(\underline{\mathrm{Hom}}_A(N_{p_0}, A)(1), A\big)(1) \cong \underline{\mathrm{Hom}}_{A^o}\big(\underline{\mathrm{Hom}}_A(N_{p_0}, A), A\big) \cong N_{p_0}.$$

By symmetry, every co-point module over $A^o$ having a linear resolution is totally reflexive, and there is a functor $\underline{\mathrm{Hom}}_{A^o}(-, A)(1) : \mathrm{clin}\, A^o \to \mathrm{clin}\, A$ such that $\underline{\mathrm{Hom}}_A(\underline{\mathrm{Hom}}_{A^o}(N, A)(1), A)(1) \cong N$ for all $N \in \mathrm{clin}\, A^o$. It follows that $\underline{\mathrm{Hom}}_A(-, A)(1) : \mathrm{clin}\, A \leftrightarrow \mathrm{clin}\, A^o : \underline{\mathrm{Hom}}_{A^o}(-, A)(1)$ is a duality. $\square$

### *3.2. The condition (**ee**)*

We will use a co-hyperplane module defined below to find algebras not satisfying (**ee**).

**Definition 3.10.** Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1, and $H = \mathbb{P}(W) \subset \mathbb{P}(V)$ a hyperplane where $W \subset V$ is a co-dimension 1 subspace. A co-hyperplane module over $A$ associated to $H$ is a cyclic graded right $A$-module defined by $L_H := A/(WA + A_{\geqslant 2}) \in \mathrm{grmod}\, A$.

There is a correspondence between hyperplane modules and co-hyperplane modules via Koszul duality.

**Lemma 3.11.** *Let $A = T(V)/(R)$ be a Koszul algebra where $R \subset V \otimes_k V$ is a subspace, and $H = \mathcal{V}(f) \subset \mathbb{P}(V^*)$ a hyperplane where $f \in A_1 = V$ is a homogeneous regular element of degree 1. If $K = A/fA \in \mathrm{lin}\, A$ is a hyperplane module over A, then $E_A(K) = L_H \in \mathrm{lin}\, A^!$ is a co-hyperplane module over $A^!$ associated to the hyperplane $H$.*

**Proof.** If $kf \subset V$ is a 1-dimensional subspace generated by $f \in A_1 = V$, then $(kf)^{\perp} := \{\lambda \in V^* \mid \lambda(f) = 0\} \subset V^*$ is a co-dimension 1 subspace, and

$$\begin{aligned} \mathcal{V}(f) &= \big\{p \in \mathbb{P}(V^*) \bigm| f(p) = 0\big\} \\ &= \big\{[\lambda] \in \mathbb{P}(V^*) \bigm| \lambda(f) = 0\big\} \\ &= \big\{[\lambda] \in \mathbb{P}(V^*) \bigm| \lambda \in (kf)^{\perp}\big\} \\ &= \mathbb{P}\big((kf)^{\perp}\big). \end{aligned}$$

Since $f \in A_1 = V$ is a homogeneous regular element of degree 1,

$$0 \to A(-1) \xrightarrow{f\cdot} A \to K \to 0$$

is a linear free resolution of $K$, so $H_K(t) = H_A(t) - tH_A(t) = (1-t)H_A(t)$. Since

$$0 \to kf \otimes_k A \to k \otimes_k A \to K \to 0$$

is the minimal free resolution of $K \in \mathrm{lin}\, A$, the minimal free resolution of $E_A(K) \in \mathrm{lin}\, A^!$ begins

$$(kf)^{\perp} \otimes_k A^! \to k^* \otimes_k A^! \to E_A(K) \to 0$$

by [18, Theorem 6.3(3)], so $E_A(K) = A^!/(kf)^{\perp}A^! \in \mathrm{lin}\, A^!$. By [18, Theorem 6.3(2)], $H_{E_A(K)}(t) = H_K(-t)/H_A(-t) = 1 + t$, so $E_A(K) = A^!/(kf)^{\perp}A^! = A^!/((kf)^{\perp}A^! + A^!_{\geqslant 2}) \in \mathrm{lin}\, A^!$ is a co-hyperplane module associated to $H = \mathcal{V}(f)$. $\square$

Let $A$ be a noetherian graded algebra and $M, N \in \operatorname{grmod} A$. Recall that if $N$ has a complete resolution $T$, then we define the Tate cohomologies by $\widehat{\underline{\operatorname{Ext}}}^i_A(N, M) := h^i(\underline{\operatorname{Hom}}_A(T, M))$. The Tate cohomologies can be defined for more general modules (see [3,8]).

**Lemma 3.12.** *Let $A = T(V)/I$ be a graded algebra finitely generated in degree 1. If $N \in \operatorname{clin} A$ is a complete co-point module with the complete co-point module sequence $\{p_n\}_{n\in\mathbb{Z}}$, and $L = L_H \in \operatorname{grmod} A$ is a co-hyperplane module associated to a hyperplane $H \subset \mathbb{P}(V)$, then $\widehat{\underline{\operatorname{Ext}}}^i_A(N, L) \neq 0$ if and only if $p_i \in H$ or $p_{i-1} \in H$.*

**Proof.** Since the complete resolution of $N \in \operatorname{clin} A$ is

$$\cdots \xrightarrow{v_2\cdot} A(-2) \xrightarrow{v_1\cdot} A(-1) \xrightarrow{v_0\cdot} A \xrightarrow{v_{-1}\cdot} A(1) \xrightarrow{v_{-2}\cdot} A(2) \xrightarrow{v_{-3}\cdot} \cdots$$

where $p_i = [v_i] \in \mathbb{P}(V)$, $\widehat{\underline{\operatorname{Ext}}}^i_A(N, L)$ are the homologies of the complex

$$\cdots \xrightarrow{\cdot v_{-3}} L(-2) \xrightarrow{\cdot v_{-2}} L(-1) \xrightarrow{\cdot v_{-1}} L \xrightarrow{\cdot v_0} L(1) \xrightarrow{\cdot v_1} L(2) \xrightarrow{\cdot v_2} \cdots.$$

Since $L = A/(WA + A_{\geqslant 2}) = k \oplus (V/W)$ where $W \subset V$ is a codimension 1 subspace such that $H = \mathbb{P}(W)$,

$$\operatorname{Ker}\{L(i) \xrightarrow{\cdot v_i} L(i+1)\} = \begin{cases} L(i) & \text{if } v_i \in W, \\ L(i)_{-i+1} & \text{if } v_i \notin W, \end{cases}$$

and

$$\operatorname{Im}\{L(i-1) \xrightarrow{\cdot v_{i-1}} L(i)\} = \begin{cases} 0 & \text{if } v_{i-1} \in W, \\ L(i)_{-i+1} & \text{if } v_{i-1} \notin W, \end{cases}$$

hence the result. $\square$

**Proposition 3.13.** *Let $A = T(V)/I$ be a noetherian AS-Gorenstein algebra with $\mathcal{P}^!(A) = (E, \sigma)$ such that $\sigma : E \to E$ is a bijection. If $N = N_p \in \operatorname{clin} A$ is a co-point module over $A$ having a linear resolution associated to a point $p \in E$, and $L = L_H \in \operatorname{grmod} A$ is a co-hyperplane module associated to a hyperplane $H \subset \mathbb{P}(V)$, then $\widehat{\underline{\operatorname{Ext}}}^i_A(N, L) \neq 0$ if and only if $\sigma^i(p) \in H$ or $\sigma^{i-1}(p) \in H$.*

**Proof.** By Theorem 3.7, for every $p \in E$, $N_p \in \operatorname{clin} A$ is a complete co-point module with the complete co-point module sequence $\{\sigma^n(p)\}_{n\in\mathbb{Z}}$, so the result follows from Lemma 3.12. $\square$

The following theorem provides more examples of algebras not satisfying (**uac**), extending [13, Corollary 6.2].

**Theorem 3.14.** *Let $A = T(V)/I$ be a noetherian AS-Gorenstein algebra with $\mathcal{P}^!(A) = (E, \sigma)$ such that $\sigma : E \to E$ is a bijection. If there are a point $p \in \mathbb{P}(V)$ and a hyperplane $H \subset \mathbb{P}(V)$ such that $0 < \#\{i \in \mathbb{Z} \mid \sigma^i(p) \in H\} < \infty$, then $A$ does not satisfy (**uac**).*

**Proof.** Let $d = \operatorname{id} A < \infty$. We fix $n \in \mathbb{Z}$ such that $\sigma^n(p) \in H$. By [14, Lemma 3.1], $\underline{\operatorname{Ext}}^i_A(N_{\sigma^{n-d-1}(p)}, L_H) \cong \widehat{\underline{\operatorname{Ext}}}^i_A(N_{\sigma^{n-d-1}(p)}, L_H)$ for $i > d$. Since $\sigma^i(\sigma^{n-d-1}(p)) = \sigma^{i+n-d-1}(p) \notin H$ for any $i \gg 0$ by the assumption, $\underline{\operatorname{Ext}}^i_A(N_{\sigma^{n-d-1}(p)}, L_H) = 0$ for all $i \gg 0$ by Proposition 3.13. However, since $\sigma^{d+1}(\sigma^{n-d-1}(p)) = \sigma^n(p) \in H$,

$$\underline{\operatorname{Ext}}^{d+1}_A(N_{\sigma^{n-d-1}(p)}, L_H) \cong \widehat{\underline{\operatorname{Ext}}}^{d+1}_A(N_{\sigma^{n-d-1}(p)}, L_H) \neq 0$$

by Proposition 3.13 again, so $A$ does not satisfy (**uac**) by [14, Corollary 3.3]. $\square$

Note that in order for the geometric condition in the above theorem to be satisfied, we must have $|\sigma| = \infty$.

The following is the main theorem of this section.

**Theorem 3.15.** *Let $A = T(V)/(R)$ be a Frobenius Koszul algebra with $\mathcal{P}^!(A) = (E, \sigma)$ such that $A^!$ is a quantum polynomial algebra, and let $\nu \in \operatorname{Aut} A$ be the Nakayama automorphism of $A$. If there are a point $p \in E$ and a hyperplane $H \subset \mathbb{P}(V)$ such that $\#\{i \in \mathbb{N} \mid \sigma^i(p) \in H\} = \infty$ but $\#\{i \in \mathbb{N} \mid \sigma^{-i}\nu(p) \in H\} < \infty$, then $A$ does not satisfy (**ee**).*

**Proof.** If $\#\{i \in \mathbb{N} \mid \sigma^i(p) \in H\} = \infty$, then $\underline{\widehat{\operatorname{Ext}}}^i_A(N, L) \neq 0$ for infinitely many $i \in \mathbb{N}$ by Proposition 3.13, so $\underline{\operatorname{Ext}}^i_A(N, L) \neq 0$ for infinitely many $i \in \mathbb{N}$ by [14, Lemma 3.1]. On the other hand, suppose that $\#\{i \in \mathbb{N} \mid \sigma^{-i}\nu(p) \in H\} < \infty$. Since

$$\underline{\widehat{\operatorname{Ext}}}^i_A(L, N) \cong \underline{\widehat{\operatorname{Ext}}}^{-1-i}_A\big((N_p)_{\nu^{-1}}(-d), L\big)^* \cong \underline{\widehat{\operatorname{Ext}}}^{-1-i}_A(N_{\nu(p)}, L)^*(-d)$$

as graded vector spaces where $d = \operatorname{gldim} A^!$ by [8, Proposition 8] and [15, Lemma 3.2(2)], $\underline{\widehat{\operatorname{Ext}}}^i_A(L, N) \neq 0$ for finitely many $i \in \mathbb{N}$ by Proposition 3.13 again, hence $\underline{\operatorname{Ext}}^i_A(L, N) \neq 0$ for finitely many $i \in \mathbb{N}$ by [14, Lemma 3.1] again. □

**Corollary 3.16.** *Let $A = T(V)/(R)$ be a Frobenius Koszul algebra with $\mathcal{P}^!(A) = (E, \sigma)$ such that $A^!$ is a quantum polynomial algebra, and let $\nu \in \operatorname{Aut} A$ be the Nakayama automorphism of $A$. If*

- *$k$ is an infinite field,*
- *$\dim_k V \geqslant 3$,*
- *$|\sigma| < \infty$, and*
- *there is a point $p \in E$ such that $\sigma^i(p) \neq \nu(p)$ for any $i \in \mathbb{Z}$,*

*then $A$ does not satisfy (**ee**).*

**Proof.** The proof is similar to that of Corollary 2.14. □

**Remark 3.17.** Recall that the map $\sigma : E \to E$ corresponds to the syzygy functor $\Omega : \underline{\operatorname{grmod}} A \to \underline{\operatorname{grmod}} A$, and the map $\nu : E \to E$ corresponds to the Nakayama functor $- \otimes_A A^* : \underline{\operatorname{grmod}} A \to \underline{\operatorname{grmod}} A$, so the condition $\sigma^i(p) \neq \nu(p)$ for any $i \in \mathbb{Z}$ in the above corollary implies that $\underline{\operatorname{grmod}} A$ is not Calabi–Yau (up to degree shifts) or $A$ is not stably symmetric in the sense of [14].

**Example 3.18.** Let $A = k\langle x, y, z\rangle/(\alpha zy + yz, \beta xz + zx, \gamma yx + xy, x^2, y^2, z^2)$ where $\alpha, \beta, \gamma \in k$ such that $\alpha\beta\gamma \neq 0, 1$. Since $A^! \cong k\langle x, y, z\rangle/(zy - \alpha yz, xz - \beta zx, yx - \gamma xy)$, $\mathcal{P}^!(A) = \mathcal{P}(A^!) = (E, \sigma)$ where $E = \mathcal{V}(xyz) \subset \mathbb{P}(V)$ and

$$\begin{aligned} \sigma(0, b, c) &= (0, \alpha b, c), \\ \sigma(a, 0, c) &= (a, 0, \beta c), \\ \sigma(a, b, 0) &= (\gamma a, b, 0) \end{aligned}$$

by [13, Theorem 3.8], and the map $\nu : E \to E$ induced by the Nakayama automorphism $\nu \in \operatorname{Aut} A$ of $A$ is given by

$$\nu(a, b, c) = \big((\gamma/\beta)a, (\alpha/\gamma)b, (\beta/\alpha)c\big)$$

by [21, Theorem 9.2]. Suppose that $k$ is an infinite field, and $\alpha, \beta, \gamma$ are roots of unity so that $|\sigma| < \infty$. If $\alpha^i \neq \beta\gamma$ for any $i \in \mathbb{Z}$, $\beta^i \neq \gamma\alpha$ for any $i \in \mathbb{Z}$, or $\gamma^i \neq \alpha\beta$ for any $i \in \mathbb{Z}$, then $A$ does not satisfy (**ee**) by the same argument as in Example 2.15.

**Remark 3.19.** In the commutative case, the condition (**uac**) implies the condition (**ee**) over noetherian commutative Gorenstein local rings [5, Theorem 4.1]. However, in the above example, if $\alpha, \beta, \gamma$ are roots of unity, then $A$ satisfies (**uac**) by [13, Theorem 6.5]. So, for example, if $\alpha = 1$ and $\beta = \gamma$ is the primitive $n$th root of unity for $n \geqslant 3$, then $A$ is a noetherian AS-Gorenstein (Frobenius) algebra satisfying (**uac**) but not satisfying (**ee**). It follows that the condition (**uac**) does not imply the condition (**ee**) over noncommutative Gorenstein rings. In [4], there is given a simple example of a noncommutative (Frobenius) algebra not satisfying (**ee**), however, that example does not satisfy (**uac**) either by [13, Theorem 6.5], so Example 3.18 may provide the first example of an algebra satisfying (**uac**) but not satisfying (**ee**). However, Example 3.18 also suggests that, for a Frobenius, non-stably symmetric Koszul algebra $A$ with $\mathcal{P}^!(A) = (E, \sigma)$, if $|\sigma| < \infty$, then $A$ tends to satisfy (**uac**) but $A$ tends not to satisfy (**ee**), so, contrary to the commutative situation, (**uac**) and (**ee**) are rather exclusive conditions.

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